Title	FIXED-WIDTH CONFIDENCE INTERVAL ESTIMATION OF A FUNCTION OF TWO EXPONENTIAL SCALE PARAMETERS (Statistical Inference of Records and Related Statistics)
Author(s)	Lim, Daisy Lou; Isogai, Eiichi; Uno, Chikara
Citation	数理解析研究所講究録 (2005), 1439: 31-40
Issue Date	2005-07
URL	http://hdl.handle.net/2433/47519
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録 1439 巻 2005 年 31-40

FIXED-WIDTH CONFIDENCE INTERVAL ESTIMATION OF A FUNCTION OF TWO EXPONENTIAL SCALE PARAMETERS

新潟大学・大学院自然科学研究科 リム ディスィー ルー (Daisy Lou Lim)
Graduate School of Science and Technology, Niigata University
新潟大学・理学部 磯貝 英一 (Eiichi Isogai)
Department of Mathematics, Niigata University
秋田大学・教育文化学部 宇野 力 (Chikara Uno)
Department of Mathematics, Akita University

1 Introduction

Many researchers are working in the area of sequential estimation in the two-sample exponential case. To cite some recent works, Mukhopadhyay and Chattopadhyay [4] considered the sequential estimation of the difference between means. Sen [5] treated a sequential comparison of two exponential distributions. Uno [6] provided second-order approximations of the expected sample size and the risk of the sequential procedure for the ratio parameter $\theta=\sigma_1/\sigma_2$. Isogai and Futschik [2] dealt with the same parameter θ, using bounded risk estimation. Lim, et al. [3], investigated the construction of sequential confidence intervals for positive functions of the scale parameters. In this paper, we will use the results of Lim, et al. [3] for the function $h(\sigma_1,\sigma_2)=(\sigma_1/\sigma_2)^r, r\neq 0$. More specifically for the cases when $r=1$ and $r=2$.

Let $h(x,y)$ be a positive, real-valued and three-times continuously differentiable function defined on $\mathbb{R}_+^2=(0,+\infty)\times(0,+\infty)$ with $h_x=\frac{\partial}{\partial x}h$, $h_y=\frac{\partial}{\partial y}h$ and $h_x^2(x,y)+h_y^2(x,y)>0$ on $\mathbb{R}_+^2$.

Let $X_1,X_2,\cdots$ and $Y_1,Y_2,\cdots$ be independent observations from two exponential populations Π_1 and Π_2, respectively, with their corresponding densities given as follows:

$$f_1(x)=\sigma_1^{-1}\exp(-x/\sigma_1)\,I(x>0)\quad\text{and}\quad f_2(y)=\sigma_2^{-1}\exp(-y/\sigma_2)\,I(y>0),$$

where the scale parameters $\sigma_1>0$ and $\sigma_2>0$ are both unknown and $I(\cdot)$ stands for the indicator function of $(\cdot)$. Taking samples of size n from Π_1 and Π_2, we estimate $\theta=h(\sigma_1,\sigma_2)$ by

$$\hat{\theta}_n=h(\bar{X}_n,\bar{Y}_n),$$

where $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$ and $\bar{Y}_n = n^{-1}\sum_{i=1}^{n} Y_i$.

Given $d > 0$ and $\alpha \in (0,1)$, we want to construct a confidence interval I_n for $\theta = h(\sigma_1, \sigma_2)$ with length $2d$ and coverage probability $1-\alpha$, based on samples of size n, $\{X_1, \cdots, X_n\}$ and $\{Y_1, \cdots, Y_n\}$, from Π_1 and Π_2, respectively. Throughout the paper, we shall assume that '$\xrightarrow{d}$', '$\xrightarrow{p}$' and '$\xrightarrow{a.s.}$' stand for convergence in distribution, convergence in probability and almost sure convergence, respectively.

Let us look at the succeeding result which gives the asymptotic distribution of $\hat{\theta}_n = h(\bar{X}_n, \bar{Y}_n)$. This result provides the asymptotic normality of $\sqrt{n}(\hat{\theta}_n - \theta)$.

Proposition 1. ([3]) *Let a function g on $\mathbb{R}_+^2$ be defined by*

$$g(x,y) = h_x^2(x,y)x^2 + h_y^2(x,y)y^2.$$

Then

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, g(\sigma_1, \sigma_2)) \quad \text{as } n \to \infty.$$

For a given $d > 0$ and $0 < \alpha < 1$, let $I_n = [\,\hat{\theta}_n - d,\ \hat{\theta}_n + d\,]$ be a confidence interval for θ with length $2d$. This interval I_n must satisfy

$$P\{\theta \in I_n\} = P\{|\hat{\theta}_n - \theta| \le d\} \ge 1 - \alpha\,. \tag{1}$$

Choose $a = a_\alpha > 0$ such that $\Phi(a) = 1 - \alpha/2$, where Φ is the standard normal distribution function. Set

$$n^* = \frac{a^2}{d^2}\, g(\sigma_1, \sigma_2)\,. \tag{2}$$

Then it follows from Proposition 1 that for all $n \ge n^*$,

$$\begin{aligned} P\{\theta \in I_n\} &= P\left\{\left|\sqrt{n}(\hat{\theta}_n - \theta)/\sqrt{g(\sigma_1,\sigma_2)}\right| \le d\sqrt{n}/\sqrt{g(\sigma_1,\sigma_2)}\right\} \\ &\ge P\left\{\left|\sqrt{n}(\hat{\theta}_n - \theta)/\sqrt{g(\sigma_1,\sigma_2)}\right| \le a\right\} \approx 1 - \alpha \end{aligned}$$

if n^* is sufficiently large. For simplicity, assume n^* to be an integer. Then n^* is the asymptotically smallest sample size which approximately satisfies equation (1).

2 Main Results

In this section, we will propose a sequential procedure and give its asymptotic properties. We have seen from the previous section that n^* in (2) is the asymptotically smallest sample size. Now, since σ_1 and σ_2 are unknown, then n^* is also unknown. It is known that fixed sample size procedures are not available for scale families. Thus, we propose the following stopping rule:

$$N = N_d = \inf\left\{ n \ge m : n \ge \frac{a^2}{d^2} g(\bar{X}_n, \bar{Y}_n) \right\}, \tag{3}$$

where $m \geq 2$ is the initial sample size. Then in view of the SLLN and the definition of N_d, we can show the lemma below.

Lemma 1. ([3])

(i) $\quad P\{N_d < +\infty\} = 1$ *for each* $d > 0$.

(ii) $\quad N_d \xrightarrow{a.s.} +\infty$ *as* $d \to 0$.

(iii) $\quad N_d/n^* \xrightarrow{a.s.} 1$ *as* $d \to 0$.

The following proposition gives the asymptotic normality of $\sqrt{N}(\hat{\theta}_N - \theta)$ which will play the important role in showing the asymptotic consistency of the sequential confidence intervals $\{I_N\}$.

Proposition 2. ([3]) *As* $d \to 0$,

$$\sqrt{N}(\hat{\theta}_N - \theta) \xrightarrow{d} N(0, g(\sigma_1, \sigma_2)),$$

where

$$g(\sigma_1, \sigma_2) = h_x^2(\sigma_1, \sigma_2)\sigma_1^2 + h_y^2(\sigma_1, \sigma_2)\sigma_2^2.$$

Once sampling is stopped after taking N observations from populations Π_1 and Π_2, respectively, we use the confidence interval $I_N = [\,\hat{\theta}_N - d,\ \hat{\theta}_N + d\,]$ for θ. The next result shows the asymptotic consistency of the sequential confidence intervals $\{I_N\}$.

Theorem 1. ([3]) [Asymptotic Consistency]

$$\lim_{d \to 0} P\{\theta \in I_N\} = 1 - \alpha.$$

Throughout the remainder of this section, we let

$$U_i = (X_i - \sigma_1)/\sigma_1, \quad V_i = (Y_i - \sigma_2)/\sigma_2 \quad \text{and} \quad \boldsymbol{X}_i = (U_i, V_i) \quad \text{for } i = 1, 2, \cdots.$$

Consider also the following notations:

$$Z_{1n} = \sqrt{n}(\bar{X}_n - \sigma_1)/\sigma_1, \quad Z_{2n} = \sqrt{n}(\bar{Y}_n - \sigma_2)/\sigma_2,$$

$$D_n = n\bar{U}_n = \sum_{i=1}^{n} U_i = n(\bar{X}_n - \sigma_1)/\sigma_1 = \sqrt{n}Z_{1n},$$

$$Q_n = n\bar{V}_n = \sum_{i=1}^{n} V_i = n(\bar{Y}_n - \sigma_2)/\sigma_2 = \sqrt{n}Z_{2n},$$

$$\boldsymbol{S}_n = (D_n, Q_n) \quad \text{and} \quad \boldsymbol{c} = \left(-\sigma_1 \frac{g_x(\sigma_1, \sigma_2)}{g(\sigma_1, \sigma_2)}, -\sigma_2 \frac{g_y(\sigma_1, \sigma_2)}{g(\sigma_1, \sigma_2)}\right).$$

Define the function f on $\mathbb{R}_+^2$ as $f(x,y) = g(\sigma_1, \sigma_2)/g(x,y)$. Since g is positive and continuous on $\mathbb{R}_+^2$, so is f. Then the stopping time N in (3) can be written as

$$N = \inf\{n \geq m : Z_n \geq n^*\}, \tag{4}$$

where

$$Z_n = nf(\bar{X}_n, \bar{Y}_n) = n - \sigma_1 \frac{g_x(\sigma_1, \sigma_2)}{g(\sigma_1, \sigma_2)} D_n - \sigma_2 \frac{g_y(\sigma_1, \sigma_2)}{g(\sigma_1, \sigma_2)} Q_n + \xi_n, \tag{5}$$

$$\xi_n = \frac{1}{2}\left\{{\sigma_1}^2 f_{xx}(\eta_1, \eta_2) Z_{1n}^2 + 2\sigma_1\sigma_2 f_{xy}(\eta_1, \eta_2) Z_{1n} Z_{2n} + {\sigma_2}^2 f_{yy}(\eta_1, \eta_2) Z_{2n}^2\right\},$$

η_1 and η_2 are random variables satisfying $|\eta_1 - \sigma_1| < |\bar{X}_n - \sigma_1|$ and $|\eta_2 - \sigma_2| < |\bar{Y}_n - \sigma_2|$. In the notations of Aras and Woodroofe [1], we can rewrite (5) as

$$Z_n = n + \langle \boldsymbol{c}, \boldsymbol{S}_n \rangle + \xi_n,$$

where $\langle \cdot, \cdot \rangle$ denotes inner product. Let

$$T = \inf\{n \geq 1 : n + \langle \boldsymbol{c}, \boldsymbol{S}_n \rangle > 0\} \quad \text{and} \quad \rho = \frac{E\{(T + \langle \boldsymbol{c}, \boldsymbol{S}_T \rangle)^2\}}{2E\{T + \langle \boldsymbol{c}, \boldsymbol{S}_T \rangle\}}. \tag{6}$$

Consider the following assumptions:

(A1) $\left\{\left[\left(Z_n - \dfrac{n}{\epsilon_0}\right)^+\right]^3, n \geq m\right\}$ is uniformly integrable for some $0 < \epsilon_0 < 1$, where $x^+ = \max(x, 0)$.

(A2) $\displaystyle\sum_{n=m}^{\infty} nP\{\xi_n < -\epsilon_1 n\} < \infty$ for some $0 < \epsilon_1 < 1$.

The following theorem gives the second-order approximation of the expected sample size $E(N)$.

Theorem 2. ([3]) *If (A1) and (A2) hold, then*

$$E(N) = n^* + \rho - \nu + o(1) \quad \text{as } d \to 0,$$

where

$$\nu = \left\{\sigma_1^2 f_{xx}(\sigma_1, \sigma_2) + \sigma_2^2 f_{yy}(\sigma_1, \sigma_2)\right\}/2$$

and ρ in (6) satisfies

$$0 < \rho < \{1 + \langle \boldsymbol{c}, \boldsymbol{c} \rangle\}/2.$$

3 Example

We consider the estimation of the rth power of the ratio of two scale parameters, namely, $\theta = h(\sigma_1, \sigma_2) = (\sigma_1/\sigma_2)^r$ for $r \neq 0$. Theorem 3 that follows, gives the expected sample size of the sequential procedure for the given function θ.

Theorem 3. *If $m > \max\{1, 6|r|\}$, then*

$$E(N) = n^* + \rho - 4r^2 + o(1) \quad \text{as } d \to 0,$$

where ρ in (6) satisfies

$$0 < \rho < \frac{1 + 8r^2}{2}.$$

Proof. For this function, the stopping random variable N in (4) can be written as

$$N = \inf\{n \geq m : Z_n \geq n^*\},$$

where

$$Z_n = n - 2r(D_n - Q_n) + \xi_n \tag{7}$$

and

$$\xi_n = r\theta^2 \left(\frac{\eta_2}{\eta_1}\right)^{2r} \left\{(2r+1)\frac{\sigma_1{}^2}{\eta_1{}^2} Z_{1n}^2 - 4r\frac{\sigma_1\sigma_2}{\eta_1\eta_2} Z_{1n}Z_{2n} + (2r-1)\frac{\sigma_2{}^2}{\eta_2{}^2} Z_{2n}^2\right\},$$

η_1 and η_2 are random variables satisfying $|\eta_1 - \sigma_1| < |\bar{X}_n - \sigma_1|$ and $|\eta_2 - \sigma_2| < |\bar{Y}_n - \sigma_2|$. In the notations of Aras and Woodroofe [1], we can rewrite (7) as

$$Z_n = n + \langle \boldsymbol{c}, \boldsymbol{S}_n \rangle + \xi_n,$$

where $\boldsymbol{c} = (-2r, 2r)$. In order to use Theorem 2 to determine the expected sample size, we need to satisfy conditions (A1) and (A2) of the theorem. Let $u > 1$ and $v > 1$ be such that $u^{-1} + v^{-1} = 1$ and M a generic positive constant.

To prove (A1), it suffices to show that

$$\sup_{n \geq m} E\left\{\left[(Z_n - n/\epsilon_0)^+\right]^3\right\} < \infty.$$

Now

$$(Z_n - n/\epsilon_0)^+ = n\left\{\left[(\bar{V}_n + 1)/(\bar{U}_n + 1)\right]^{2r} - \epsilon_0^{-1}\right\} I_{\left\{\left[(\bar{V}_n+1)/(\bar{U}_n+1)\right]^{2r} > \epsilon_0^{-1}\right\}}.$$

Thus,

$$E\left\{\left[(Z_n - n/\epsilon_0)^+\right]^3\right\} \leq n^3 E\left\{\left[(\bar{V}_n + 1)/(\bar{U}_n + 1)\right]^{6r} I_{\left\{\left[(\bar{V}_n+1)/(\bar{U}_n+1)\right]^{2r} > \epsilon_0^{-1}\right\}}\right\}$$

$$\leq n^3 E\left\{\left[(\bar{V}_n + 1)/(\bar{U}_n + 1)\right]^{6r} I_{\left\{\left[(\bar{V}_n+1)/(\bar{U}_n+1)\right]^{2r} > \epsilon_0^{-1},\, \bar{U}_n+1 < 1-\epsilon_0\right\}}\right\}$$

$$+ n^3 E\left\{\left[(\bar{V}_n + 1)/(\bar{U}_n + 1)\right]^{6r} I_{\left\{\left[(\bar{V}_n+1)/(\bar{U}_n+1)\right]^{2r} > \epsilon_0^{-1},\, \bar{U}_n+1 \geq 1-\epsilon_0\right\}}\right\}$$

$$\equiv K_1(n) + K_2(n), \text{ say.}$$

By the independence of $\bar{U}_n$ and $\bar{V}_n$ and by Hölder's Inequality, we have $K_1(n) \leq n^3 E(\bar{V}_n+1)^{6r}\left\{E(\bar{U}_n+1)^{-6ru}\right\}^{1/u}\left\{P(|\bar{U}_n|>\epsilon_0)\right\}^{1/v}$. By Lemma 1 of Uno [6], $E(\bar{V}_n+1)^{6r} \leq M$ and $E(\bar{U}_n+1)^{-6ru} \leq M$ for $n \geq m > 6|r|u$. By Markov's Inequality, $P(|\bar{U}_n|>\epsilon_0) \leq (n\epsilon_0)^{-q}E|D_n|^q$ for $q \geq 2$. But by Marcinkiewicz-Zygmund Inequality, $E|D_n|^q = O(n^{q/2})$ as $n \to \infty$. Thus, it follows that $K_1(n) \leq Mn^{3-q/2v}$ for $n \geq m > 6|r|u$. Since $m > 6|r|$, we can choose $u > 1$ such that $m > 6|r|u$. Then choose $q > \max\{2, \frac{6u}{u-1}\}$. Thus, $3 - q/2v \leq 0$ which shows that $\sup_{n\geq m} K_1(n) < \infty$. Let $\delta = \epsilon_0{}^{-1/2r}(1-\epsilon_0) > 1$ and $r > 0$ for small $0 < \epsilon_0 < 1$. Then

$$\left\{\left[(\bar{V}_n+1)/(\bar{U}_n+1)\right]^{2r} > \epsilon_0^{-1},\ \bar{U}_n+1 \geq 1-\epsilon_0\right\} \subset \left\{\bar{V}_n+1 \geq \delta\right\}.$$

It follows that for $r > 0$,

$$\begin{aligned}
K_2(n) &\leq n^3(1-\epsilon_0)^{-6r}E\left\{(\bar{V}_n+1)^{6r}I_{\{\bar{V}_n+1\geq\delta\}}\right\} \\
&\leq n^3(1-\epsilon_0)^{-6r}\left\{E(\bar{V}_n+1)^{6ru}\right\}^{1/u}\left\{P(\bar{V}_n+1\geq\delta)\right\}^{1/v} \\
&\leq n^3(1-\epsilon_0)^{-6r}\left\{E(\bar{V}_n+1)^{6ru}\right\}^{1/u}\left\{P(|\bar{V}_n|\geq\delta-1)\right\}^{1/v},
\end{aligned}$$

where $\frac{1}{u}+\frac{1}{v}=1$ and $u > 1$. Thus, in the same way as $K_1(n)$, $\sup_{n\geq m} K_2(n) < \infty$ for $m > 6r$. For $r < 0$, by similar arguments as above, $\sup_{n\geq m} K_2(n) < \infty$ for $m > 6|r|$. This completes the proof of (A1).

By Taylor's Theorem,

$$\begin{aligned}
&(\bar{V}_n+1)^{2r}(\bar{U}_n+1)^{-2r} \\
&= \left(1+2r\bar{V}_n+r(2r-1)\phi_2^{2(r-1)}\bar{V}_n^2\right)\left(1-2r\bar{U}_n+r(2r+1)\phi_1^{-2(r+1)}\bar{U}_n^2\right),
\end{aligned}$$

where ϕ_1 and ϕ_2 are positive random variables between $(\bar{U}_n+1)$ and 1, and $(\bar{V}_n+1)$ and 1, respectively. Thus, it follows from (7) that

$$\begin{aligned}
\xi_n &= Z_n - n + 2r(D_n - Q_n) = n\left[(\bar{V}_n+1)^{2r}(\bar{U}_n+1)^{-2r} - 1 + 2r(\bar{U}_n - \bar{V}_n)\right] \\
&= n\left[-4r^2\bar{U}_n\bar{V}_n + r(2r+1)\phi_1^{-2(r+1)}\bar{U}_n^2 + 2r^2(2r+1)\phi_1^{-2(r+1)}\bar{U}_n^2\bar{V}_n\right] \\
&\quad + n\left[r(2r-1)\phi_2^{2(r-1)}\bar{V}_n^2 - 2r^2(2r-1)\phi_2^{2(r-1)}\bar{U}_n\bar{V}_n^2\right. \\
&\quad \left. + r^2(4r^2-1)\phi_1^{-2(r+1)}\phi_2^{2(r-1)}\bar{U}_n^2\bar{V}_n^2\right].
\end{aligned}$$

Thus, setting $\epsilon_2 = \epsilon_1/6$ for $0 < \epsilon_1 < 1$, we have

$$
\begin{aligned}
&P\{\xi_n < -\epsilon_1 n\} \\
&\le P\left\{\left|4r^2\bar{U}_n\bar{V}_n\right| > \epsilon_2\right\} + P\left\{\left|r(2r+1)\phi_1^{-2(r+1)}\bar{U}_n^2\right| > \epsilon_2\right\} \\
&\quad + P\left\{\left|2r^2(2r+1)\phi_1^{-2(r+1)}\bar{U}_n^2\bar{V}_n\right| > \epsilon_2\right\} + P\left\{\left|r(2r-1)\phi_2^{2(r-1)}\bar{V}_n^2\right| > \epsilon_2\right\} \\
&\quad + P\left\{\left|2r^2(2r-1)\phi_2^{2(r-1)}\bar{U}_n\bar{V}_n^2\right| > \epsilon_2\right\} \\
&\quad + P\left\{\left|r^2(4r^2-1)\phi_1^{-2(r+1)}\phi_2^{2(r-1)}\bar{U}_n^2\bar{V}_n^2\right| > \epsilon_2\right\} \\
&\equiv \sum_{i=1}^{6} I_i(n), \text{ say.}
\end{aligned}
$$

By the independence of $\bar{U}_n$ and $\bar{V}_n$, and by Marcinkiewicz-Zygmund Inequality, $E\{|D_nQ_n|^q\} = E\{|D_n|^q\}E\{|Q_n|^q\} \le Mn^q$, for $q \ge 2$. Thus, by Markov's Inequality,

$$
I_1(n) = P\left\{4r^2|D_nQ_n| > n^2\epsilon_2\right\} \le M\,n^{-2q}E\{|D_nQ_n|^q\} \le Mn^{-q}.
$$

Now, since ϕ_1 is a random variable between 1 and $\bar{U}_n + 1$, then $\phi_1 > 1/2$ on the set $\{|\bar{U}_n| \le 1/4\}$. Thus, for $r+1 \ge 0$, we have

$$
\begin{aligned}
I_2(n) &\le P\left\{M\left|\phi_1^{-2(r+1)}\bar{U}_n^2\right| > \epsilon_2,\, |\bar{U}_n| \le 1/4\right\} + P\{|\bar{U}_n| > 1/4\} \\
&\le P\left\{M(1/2)^{-2(r+1)}(1/2)^2|\bar{U}_n| > \epsilon_2\right\} + P\{|\bar{U}_n| > 1/4\} \\
&\le P\left\{|\bar{U}_n| > M\right\} + Mn^{-q/2} \le Mn^{-q/2}.
\end{aligned}
$$

In a similar way, we get

$$
\begin{aligned}
I_3(n) &\le P\left\{M\left|\phi_1^{-2(r+1)}\bar{U}_n^2\bar{V}_n\right| > \epsilon_2,\, |\bar{U}_n| \le 1/4\right\} + P\{|\bar{U}_n| > 1/4\} \\
&\le P\left\{|\bar{U}_n\bar{V}_n| > M\right\} + Mn^{-q/2} \\
&\le Mn^{-2q}E\{|D_nQ_n|^q\} + Mn^{-q/2} \le Mn^{-q/2}.
\end{aligned}
$$

Suppose that $r+1 < 0$. Then it follows by convexity and Lemma 1 of Uno [6] that for any $q \ge 2$

$$
E\left\{\phi_1^{-4(r+1)q}\right\} \le 1 + E\left[(\bar{U}_n+1)^{-4(r+1)q}\right] \le M.
$$

Thus,

$$
I_2(n) \le ME\left\{\phi_1^{-4(r+1)q}\right\}^{1/2} E\left\{|\bar{U}_n|^{4q}\right\}^{1/2} \le Mn^{-q}. \tag{8}
$$

From (8), we obtain

$$
\begin{aligned}
I_3(n) &\le ME\left\{\left|\phi_1^{-2(r+1)}\bar{U}_n^2\bar{V}_n\right|^q\right\} = ME\left\{\left|\phi_1^{-2(r+1)}\bar{U}_n^2\right|^q\right\}E\{|\bar{V}_n|^q\} \\
&\le Mn^{-q}n^{-q/2} \le Mn^{-q/2}.
\end{aligned}
$$

Thus, from the above relations, $I_i(n) \leq Mn^{-q/2}$ for $i = 1, 2, 3$. Hence, taking $q = 6$, we have $\sum_{n=1}^{\infty} nI_i(n) < \infty$ for $i = 1, 2, 3$. By similar arguments, we can show that $\sum_{n=1}^{\infty} nI_i(n) < \infty$ for $i = 4, 5, 6$. Therefore, (A2) is satisfied. Now, $\nu = 4r^2$. Hence, it follows from Theorem 2 that for $m > \max\{1, 6|r|\}$,

$$E(N) = n^* + \rho - 4r^2 + o(1) \quad \text{as } d \to 0,$$

where $0 < \rho < (1 + 8r^2)/2$. This completes the proof. $\square$

To illustrate these results, let us consider two cases. For the case when $r = 1$, we consider two stopping rules; N in (3) and N^* given in Isogai and Futschik [2], and compare the coverage probabilities of the sequential confidence intervals, corresponding to N and N^*. The stopping rule N becomes

$$N = N_d = \inf\left\{n \geq m :\ n \geq \frac{2a^2\overline{X}_n^2}{d^2\overline{Y}_n^2}\right\}.$$

Then, letting $L(n) \equiv 1$ and replacing w by d^2/a^2, N in (4) is the same as N_w in Isogai and Futschik [2] who also showed that (A1) and (A2) hold with $m > 6$ and $\boldsymbol{c} = (-2, 2)$. Thus, it follows from Theorem 2 that

$$E(N) = n^* + \rho - 4 + o(1) \quad \text{and} \quad 0 < \rho < 9/2.$$

By simulation, we can get $\rho = 2.03$. Thus, taking this ρ into account, we consider another stopping rule:

$$N^* = N_d^* = \inf\left\{n \geq m :\ n \geq L(n)\frac{2a^2\overline{X}_n^2}{d^2\overline{Y}_n^2}\right\} \quad \text{where } L(n) = 1 + \frac{1.97}{n}.$$

From Theorem 2.1 of Isogai and Futschik [2], if $m > 6$ then $E(N^*) = n^* + o(1)$ as $d \to 0$.

Now, from Proposition 2.1 of Isogai and Futschik [2] if $m > 12$, then

$$E(\hat{\theta}_N) - \theta = -\frac{3d}{a\sqrt{2n^*}} + o(d^2) \quad \text{as } d \to 0.$$

From this result, we propose the following bias-corrected sequential confidence intervals:

$$I_N^\dagger = [\,\hat{\theta}_N^\dagger - d,\ \hat{\theta}_N^\dagger + d\,] \quad \text{and} \quad I_{N^*}^\dagger = [\,\hat{\theta}_{N^*}^\dagger - d,\ \hat{\theta}_{N^*}^\dagger + d\,],$$

where $\hat{\theta}_N^\dagger = \hat{\theta}_N + (3d)/(a\sqrt{2N})$ and $\hat{\theta}_{N^*}^\dagger = \hat{\theta}_{N^*} + (3d)/(a\sqrt{2N^*})$.

For the case when $r = 2$, the stopping rule in (3) becomes

$$N = N_d = \inf\left\{n \geq m :\ n \geq \frac{8a^2\overline{X}_n^4}{d^2\overline{Y}_n^4}\right\},$$

and by Theorem 3, for $m > 12$, the expected sample size is

$$E(N) = n^* + \rho - 16 + o(1) \quad \text{and} \quad 0 < \rho < 33/2.$$

Now, by simulation using 100,000 repetitions, we can get $\rho = 4.02$. Considering this value for ρ, we propose another stopping rule as follows:

$$N^* = N_d^* = \inf\left\{ n \geq m : \; n \geq L(n) \frac{8a^2 \overline{X}_n^4}{d^2 \overline{Y}_n^4} \right\}, \quad L(n) = 1 + \frac{11.98}{n}.$$

Simulation Results. We shall give simulation results for the case when $(\sigma_1, \sigma_2) = (2, 1)$. The coverage probability is set at $1 - \alpha = 0.95$ and the pilot sample size at $m = 13$. The following results are based on 10,000 repetitions.

Table 1.1 Using N ($r = 1$) $\theta = 2$

n^*	20	100	200	500	1000
d	1.239588	0.554360	0.391992	0.247918	0.175304
$E(N)$	21.4789	96.7799	197.5324	497.2630	995.8871
$E(\hat{\theta}_N)$	1.865092	1.917183	1.965773	1.986306	1.991834
$E(\hat{\theta}_N^\dagger)$	2.172344	1.981733	1.996558	1.998415	1.997865
$P(\theta \in I_N)$	0.9864	0.9079	0.9361	0.9477	0.9485
$P(\theta \in I_N^\dagger)$	0.9878	0.9241	0.9444	0.9501	0.9518

Table 1.2. Using N^* ($r = 1$) $\theta = 2$

n^*	20	100	200	500	1000
d	1.239588	0.554360	0.391992	0.247918	0.175304
$E(N^*)$	22.7216	98.7350	199.7856	499.7327	1000.3485
$E(\hat{\theta}_{N^*})$	1.860984	1.920277	1.967711	1.987303	1.994298
$E(\hat{\theta}_{N^*}^\dagger)$	2.160043	1.983678	1.998279	1.999382	2.000316
$P(\theta \in I_{N^*})$	0.9881	0.9122	0.9360	0.9460	0.9478
$P(\theta \in I_{N^*}^\dagger)$	0.9883	0.9271	0.9437	0.9476	0.9509

Table 2.1. Using N ($r = 2$) $\theta = 4$

n^*	20	100	200	500	1000
d	4.958350	2.217442	1.567968	0.991670	0.701217
$E(N)$	23.7237	83.0451	173.9840	475.8522	980.8059
$E(\hat{\theta}_N)$	3.489305	3.288380	3.492996	3.812448	3.924260
$P(\theta \in I_N)$	0.9992	0.8055	0.8122	0.8973	0.9298

Table 2.2 Using N^* $(r = 2)$ $\theta = 4$

n^*	20	100	200	500	1000
d	4.958350	2.217442	1.567968	0.991670	0.701217
$E(N^*)$	29.0472	98.0366	192.7931	493.2553	996.6286
$E(\hat{\theta}_{N^*})$	3.448231	3.438694	3.623196	3.853191	3.939215
$P(\theta \in I_{N^*})$	0.9994	0.8556	0.8657	0.9184	0.9378

The tables show that the rate of convergence of the coverage probability $P(\theta \in I_N)$ to $1 - \alpha$ seems to be slow. For the case when $r = 1$, the bias-corrected sequential confidence intervals, $I_N^\dagger$ and $I_{N^*}^\dagger$, are more effective than the ordinary ones, I_N and I_{N^*}. Furthermore, there seems to be no significant difference between the coverage probabilities of the intervals, I_N and I_{N^*}. An improvement on the stopping rule in (4) is needed.

References

[1] G. Aras and M. Woodroofe, Asymptotic expansions for the moments of a randomly stopped average, Ann. Statist., 21 (1993), 503-519.

[2] E. Isogai and A. Futschik, Sequential point estimation of the ratio of two exponential scale parameters, Sri Lankan Statistical Conference Proceedings - 2004, B.M. de Silva and N. Mukhopadhyay eds. 2004, 263-275.

[3] D.L. Lim, E. Isogai and C. Uno, Two-sample fixed width confidence intervals for a function of exponential scale parameters, Far East J. Theo. Stat. 14(2) (2004), 215-227.

[4] N. Mukhopadhyay and S. Chattopadhyay, Sequential methodologies for comparing exponential mean survival times, Sequential Analysis, 10 (1991), 139-148.

[5] P. K. Sen, Section 28.6 of The Exponential Distribution, Theory, Methods and Applications edited by N. Balakrishnan and A. Basu, Gordon and Breach Science Publishers SA, 1995.

[6] C. Uno, Sequential estimation of the ratio of scale parameters in the exponential two-sample problem, J. Japan Statist. Soc., 33 (2003), 231-244.